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PISA 2021 MATHEMATICS FRAMEWORK (FIRST DRAFT)

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Andreas Schleicher, Director for Education and Skills and Special Advisor on Education Policy to OECD's Secretary-General (andreas.schleicher@oecd.org)

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PISA 2021 Mathematics Framework (First Draft)

1. This document is a first draft of the PISA 2021 mathematics framework. It was prepared by the expert group for mathematics under the guidance of RTI International as the international contractor who leads this work. It builds on the advice provided by the Strategic Advisory Group for mathematics that worked during 2017, as well as the guidance from the PGB at the 44th meeting in Paris in November 2017. The framework will go through several additional rounds of revision and updates between now and 2020. A second draft will be finalised in autumn 2018, taking into account the PGB's feedback on the first draft.

2. The PGB is invited to:

- **COMMENT** on the first draft framework, and
- **PROVIDE DIRECTIONS** for its further development.

3. Before the framework is finalised, the following will be added:

- Illustrative examples and references to those illustrative examples for mathematical reasoning; each of the problem solving processes; and the four mathematical content domains.
- An Annex that provides a detailed exposition on the relationship between mathematics, mathematical literacy and computational reasoning.
- Relevant paragraphs on the *Structure of the Survey Instrument*.
- Figure 3 will be replaced with a screen shot of a CBAM simulation when that becomes available.
- Proficiency scale figures for mathematical literacy will be added after the data collection.
- Paragraph 17 will be expanded to better describe the different people and organisations that have played a role in the development of the framework.

INTRODUCTION

4. The assessment of mathematics has particular significance for PISA 2021, as mathematics is again the major domain assessed. Although mathematics was assessed by PISA in 2000, 2003, 2006, 2009, 2012, 2015 and 2018, the domain was the main area of focus only in 2003 and 2012.

5. The return of mathematics as the major domain in PISA 2021 provides both the opportunity to continue to make comparisons in student performance over time, and to re-examine what should be assessed in light of changes that have occurred in the world, the field and in instructional policies and practices.

6. Each country has a vision of mathematical competence and organizes their schooling to achieve it as an expected outcome. Mathematical competence historically encompassed performing basic arithmetic skills or operations, including adding, subtracting, multiplying, and dividing whole numbers, decimals, and fractions; computing percentages; and computing the area and volume of simple geometric shapes. In recent times, the digitisation of many aspects of life, the ubiquity of data for making personal decisions involving health and investments, as well as major societal challenges to address areas such as climate change, taxation, governmental debt, population growth, spread of pandemic diseases and the global economy, have reshaped what it means to be mathematically competent and to be well equipped to participate as a thoughtful, engaged, and reflective citizen in the 21st century.

7. The critical issues listed above as well as others that are facing societies throughout the world all have a quantitative component to them. Understanding them, as well as addressing them, at least in part, requires thinking mathematically. Such thinking in more and more complex contexts is not driven by the reproduction of the basic computational procedures mentioned earlier, but rather by reasoning, and the important role of reasoning demands a reconsideration of what it means for students to be competent in mathematics. Mathematical literacy goes beyond problem solving, to a deeper level, that of mathematical reasoning and computational thinking, which provides the intellectual acumen behind problem solving in the 21st century.

8. Countries today face new opportunities and challenges in all areas of life, many of which stem from the rapid deployment of computers and computational devices like robots, smartphones and networked machines. For example, the vast majority of young adults and students who started university post 2015 have always considered phones to be mobile hand-held devices capable of sharing voice, texts, and images and accessing the internet – capabilities seen as science fiction by many of their parents and certainly by all of their grandparents (Beloit College, 2017[1]). The recognition of the growing contextual discontinuity between the last century and the future has prompted a discussion around the development of 21st century skills in students (Ananiadou and Claro, 2009[2]; Fadel, Bialik and Trilling, 2015[3]; National Research Council, 2012[4]; Reimers and Chung, 2016[5]).

9. It is this discontinuity that also drives the need for education reform and the challenge of achieving it. Periodically, educators, policy makers, and other stakeholders revisit public education standards and policies. In the course of these deliberations new or revised responses to two general questions are generated: 1) What do students need to learn, and 2) Which students need to learn what? The most used argument in defence of common mathematics education for all students is its usefulness in various practical situations. However, this argument alone gets weaker with time – a lot of simple activities have been automated. Not so long ago waiters in restaurants would multiply and add on paper to calculate the price to be paid. Today they just press buttons on hand-held devices. Not so long ago people used printed timetables to plan travel – it required a good understanding of the time axis and inequalities. Today we just make a direct internet inquiry.

10. As to the question of "what to teach", many misunderstandings arise from the way mathematics is conceived. Many people see mathematics as no more than a useful toolbox. A clear trace of this approach can be found in school curricula in many countries. These are sometimes confined to a list of mathematics topics or procedures, with students asked to practice a selected few, in predictable (often test) situations. This perspective on mathematics is far too narrow for today's world. It overlooks key features of mathematics that are growing in importance.

11. Ultimately the answer to these questions is that every student should learn (and be given the opportunity to learn) to think mathematically, using mathematical reasoning in conjunction with a small set of fundamental mathematical concepts that support this reasoning and which themselves are not necessarily taught explicitly but are made manifest and reinforced throughout a student's learning experiences in mathematics. This equips students with a conceptual framework through which to address the quantitative dimensions of life in the 21st century. This is the new mathematical literacy which includes problem solving but goes beyond it to being mathematically competent.

12. The PISA 2021 framework is designed to make the relevance of mathematics to 15-year-old students clearer and more explicit, while ensuring that the items developed remain set in meaningful and authentic contexts. The mathematical modelling cycle, used in earlier frameworks (e.g. OECD (2004[6]; 2013[7])) to describe the stages individuals go through in solving contextualised problems, remains a key feature of the PISA 2021 framework. It is used to help define the mathematical processes in which students engage as they solve problems – processes that together with reasoning will provide the primary reporting dimensions.

13. For PISA 2021, computer-based assessment of mathematics (CBAM) will be the primary mode of delivery for assessing mathematical literacy. However, paper-based assessment instruments will be provided for countries choosing not to test their students by computer. The framework has been updated to also reflect the change in delivery mode introduced in 2015, including a discussion of the considerations that should inform the development of the CBAM items as this will be the first major update to the mathematics framework since computer-based assessment was introduced in PISA.

14. The development of the PISA 2021 framework takes into account the expectation of OECD that there will be an increase in the participation in PISA of low- and middle-income countries. In particular the PISA 2021 framework recognises the need to increase the resolution of the PISA assessments at the lower end of the student performance distribution by drawing from the PISA for Development (OECD, 2017[8]) framework when developing the assessment; the need to expand the performance scale at

the lower end; the importance of capturing a wider range of social and economic contexts; and the anticipation of incorporating an assessment of out-of-school 14- to 16-year-olds.

15. The increasing and evolving presence of computers and computing tools in both day-to-day life and in doing mathematics is reflected in the recognition in the PISA 2021 framework that students should possess and be able to demonstrate computational thinking skills as part of their problem-solving practice. Computational thinking skills include pattern recognition, decomposition, determining which (if any) computing tools could be employed in analysing or solving a problem, and defining algorithms as part of a detailed solution. By foregrounding the importance of computational thinking, the framework anticipates a reflection by participating countries on the role of computational thinking in mathematics curricula and pedagogy.

16. The PISA 2021 mathematics framework is organised into three major sections. The first section, 'Definition of Mathematical Literacy', explains the theoretical underpinnings of the PISA mathematics assessment, including the formal definition of the *mathematical literacy* construct. The second section, 'Organisation of the Domain', describes four aspects: a) mathematical reasoning and the three mathematical *processes* (of the modelling/problem solving cycle); b) the way mathematical *content* knowledge is organised in the PISA 2021 framework, and the content knowledge that is relevant to an assessment of 15-year-old students; c) the relationship between mathematical literacy and the so-called *21st Century skills*; and d) the *contexts* in which students will face mathematical challenges. The third section, 'Assessing Mathematical Literacy', outlines structural issues about the assessment, including a test blueprint and other technical information.

17. The 2021 framework was written under the guidance of the 2021 mathematics expert group (MEG), a body appointed by the main PISA contractors in consultation with the PISA Governing Board (PGB). The MEG members included mathematicians, mathematics educators, and experts in assessment, technology, and education research from a range of countries.

DEFINITION OF MATHEMATICAL LITERACY

18. An understanding of mathematics is central to a young person's preparedness for participation in and contribution to modern society. A growing proportion of problems and situations encountered in daily life, including in professional contexts, require some level of understanding of mathematics before they can be properly understood and addressed. Mathematics is a critical tool for young people as they confront a wide range of issues and challenges in the various aspects of their lives.

19. It is therefore important to have an understanding of the degree to which young people emerging from school are adequately prepared to use mathematics to think about their lives, plan their futures and reason about and solve meaningful problems related to a range of important issues in their lives. An assessment at age 15 provides countries with an early indication of how individuals may respond in later life to the diverse array of situations they will encounter that both involve mathematics and rely on mathematical reasoning and problem solving to make sense of.

20. As the basis for an international assessment of 15-year-old students, it is reasonable to ask: "What is important for citizens to know and be able to do in situations that involve mathematics?" More specifically, what does being mathematically competent mean for a 15-year-old, who may be emerging from school or preparing to pursue more specialised training for a career or university admission? It is important that the construct of mathematical literacy, which is used in this framework to denote the capacity of individuals to reason mathematically and solve problems in a variety of 21st century contexts, not be perceived as synonymous with minimal, or low-level, knowledge and skills. Rather, it is intended to describe the capacities of individuals to reason mathematically and use mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. This conception of mathematical literacy supports the importance of students developing a strong understanding of concepts of pure mathematics and the benefits of being engaged in explorations in the abstract world of mathematics. The construct of mathematical literacy, as defined for PISA, strongly emphasises the need to develop students' capacity to use mathematics in context, and it is important that they have rich experiences in their mathematics classrooms to accomplish this. This is as true for those 15-year-old students who are close to the end of their formal mathematics training, students who will continue with the formal study of mathematics, as well as out of school 15-year-olds. In addition, it can be argued that for almost all students, the motivation to learn mathematics increases when they see the relevance of what they are learning to both the world outside the classroom and to other subjects.

21. Mathematical literacy transcends age boundaries. For example, OECD's Programme for the International Assessment of Adult Competencies (PIAAC) defines numeracy as *the ability to access, use, interpret, and communicate mathematical information and ideas, in order to engage in and manage the mathematical demands of a range of situations in adult life.* The parallels between this definition for adults

and the PISA 2021 definition of mathematical literacy for 15-year-olds are both marked and unsurprising.

22. The assessment of mathematical literacy for 15-year-olds must take into account relevant characteristics of these students; hence, there is a need to identify age-appropriate content, language and contexts. This framework distinguishes between broad categories of content that are important to mathematical literacy for individuals generally, and the specific content topics that are appropriate for 15-year-old students. Mathematical literacy is not an attribute that an individual either has or does not have. Rather, mathematical literacy is an attribute that is on a continuum, with some individuals being more mathematically literate than others – and with the potential for growth always present.

23. For the purposes of PISA 2021, mathematical literacy is defined as follows:

> *Mathematical literacy is an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective 21st century citizens.*

24. The PISA 2021 framework, when compared with the PISA 2003 and PISA 2012 frameworks, acknowledges a number of shifts in the world of the student which in turn signal a shift on how to assess mathematical literacy in comparison to the approach used in previous frameworks. The trend away from the need to perform basic calculations to a world in which citizens are creative and engaged, making judgements for themselves and the society in which they live, in a rapidly changing world prompted by new technologies and trends.

25. As technology plays a growing role in the lives of students, the long-term trajectory of mathematical literacy should also encompass the synergistic and reciprocal relationship between mathematical thinking and computational thinking, regarded as a thought process entailed in formulating problems and designing their solutions in a form that can be executed by a computer, a human, or a combination of both (Cuny, Snyder and Wing, 2010[9]). The roles computational thinking plays in mathematics include how specific mathematical topics interact with specific computing topics, and how mathematical reasoning complements computational thinking reasoning (Gadanidis, 2015[10]; Rambally, 2017[11]). For example, Pratt and Noss (2002[12]) discuss the use of a computational microworld for developing mathematical knowledge in the case of randomness and probability; Gadanidis et al. (2018[13]) propose an approach to engage young children with ideas of group theory, using a combination of hands-on and computational thinking tools. Hence, while mathematics education evolves in terms of the tools available and the potential ways to support students in exploring the powerful ideas of the discipline (Pei, Weintrop and Wilensky, 2018[14]), the thoughtful use of computational thinking tools and skill sets can deepen the learning of mathematics contents by creating effective learning conditions (Weintrop et al., 2016[15]). Moreover, computational thinking tools offer students a context in which they can reify abstract constructs (by exploring and engaging with maths concepts in a dynamic way), as well as express ideas in new ways and interact with concepts through media and new representational tools (Grover, 2018[16]; Niemelä et al., 2017[17]; Pei, Weintrop and Wilensky, 2018[14]; Resnick et al., 2009[18]).

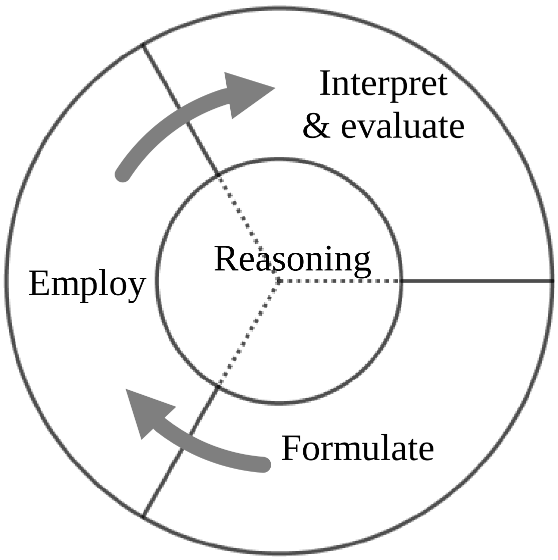
A View of Mathematically Literate Individuals in PISA 2021

26. The focus of the language in the definition of mathematical literacy is on active engagement with mathematics, and is intended to encompass mathematical reasoning and problem solving using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena.

27. It is important to note that the definition not only focuses on the use of mathematics to solve real-world problems, but also identifies mathematical reasoning as a core aspect of mathematical literacy. The contribution that the PISA 2021 framework makes is to highlight the centrality of mathematical reasoning both to the problem solving cycle and to mathematical literacy in general.

28. ***Figure 1*** depicts the relationship between mathematical reasoning and problem solving as reflected in the mathematical modelling cycle of both the PISA 2003 and PISA 2012 framework. The figure depicts the relationship between reasoning and problem solving both in general and for the PISA 2021 framework in particular.

Figure 1. Mathematical literacy: the relationship between mathematical reasoning and the problem solving (modelling) cycle.



29. In order for students to be mathematically literate they must be able, first to use their mathematics content knowledge to recognise the mathematical nature of a situation (problem) including those situations encountered in the real world and then to formulate it in mathematical terms. This transformation – from an ambiguous, messy, real-world situation to a well-defined mathematics problem – requires mathematical reasoning and is, perhaps, the critical component of what it means to be mathematically literate. Once the transformation is successfully made, the resulting mathematical problem needs to be solved using the mathematics concepts, algorithms and procedures taught in schools. However, it may require the making of strategic decisions about the selection of those tools and the order of their application – this is also a manifestation

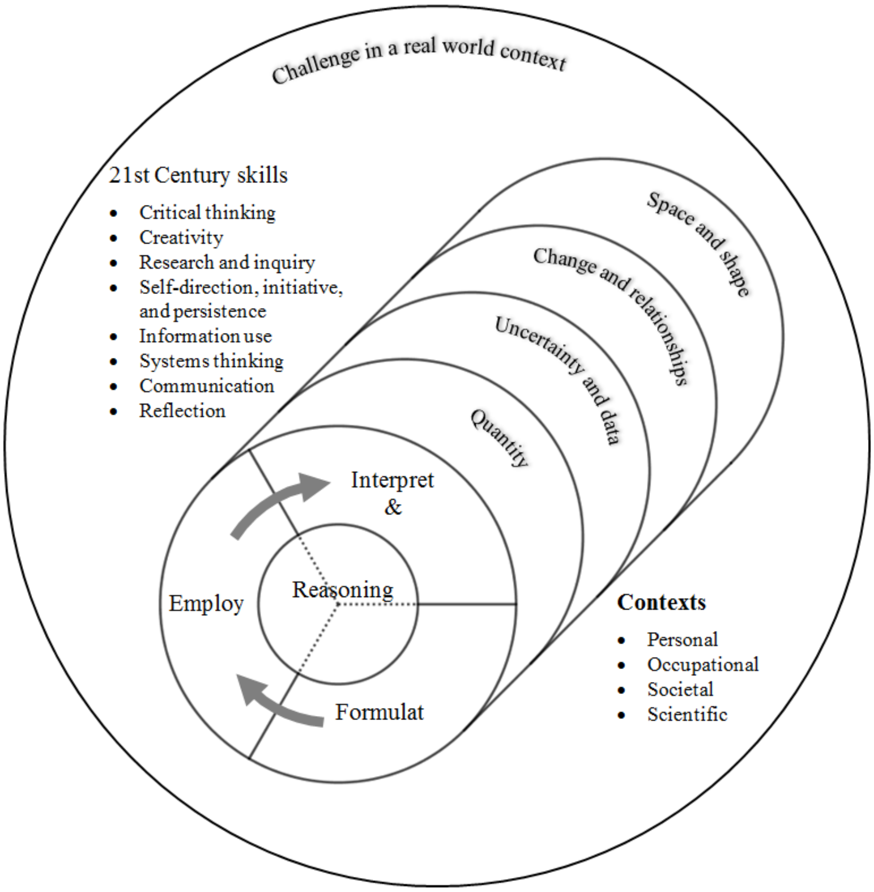
of mathematical reasoning. Finally, the PISA definition reminds us of the need for the student to evaluate the mathematical solution by interpreting the results within the original real-world situation. Additionally, students should also possess and be able to demonstrate computational thinking skills as part of their problem-solving practice. These skills include pattern recognition, decomposition, determining which (if any) computing tools could be employed in the analysing or solving the problem, and defining algorithms as part of a detailed solution.

30. Although mathematical reasoning and solving real-world problems overlap, there is an aspect to mathematical reasoning which goes beyond solving practical problems; it is also a way of evaluating and making arguments, interpretations and inferences related to important public policy debates that are, by their quantitative nature, best understood mathematically.

31. Mathematical literacy therefore comprises two related aspects: *mathematical reasoning* and *problem solving*. *Mathematical literacy* plays an important role in being able to use mathematics to *solve real-world problems*. However, mathematical reasoning also goes beyond solving problems in the traditional sense to include making informed judgements in general about that important family of societal issues which can be addressed mathematically. It also includes making judgements about the validity of information that bombards individuals by means of considering their quantitative and logical, implications. It is here where mathematical reasoning also contributes to the development of a select set of 21st century skills (discussed elsewhere in the framework).

32. The outer circle of Figure 2 shows that mathematical literacy takes place in the context of a challenge or problem that arises in the real world.

Figure 2. PISA 2021: the relationship between mathematical reasoning, the problem solving (modelling) cycle, mathematical contents, context and selected 21st century skills.



33. Figure 2 also depicts the relationship between mathematical literacy as depicted in Figure 1 and: the mathematical contents domains in which mathematical literacy is applied; the problem contexts and the selected 21st century skills that are both supportive of and developed through mathematical literacy.

34. These categories of mathematics content include: quantity, uncertainty and data, change and relationships, and space and shape. It is these categories of mathematics content knowledge which students must draw on to reason, to formulate the problem (by transforming the real world situation into a mathematical problem situation), to solve the mathematical problem once formulated, and to interpret and evaluate the solution determined.

35. As in the previous frameworks, the four context areas that PISA continues to use to define real-world situations are personal, occupational, societal and scientific. The context may be of a personal nature, involving problems or challenges that might confront an individual or one's family or peer group. The problem might instead be set in a societal context (focusing on one's community – whether it be local, national or global), an occupational context (centred on the world of work), or a scientific context (relating to the application of mathematics to the natural and technological world).

36. Included for the first time in the PISA 2021 framework (and depicted in Figure 2) are selected 21st century skills that mathematical literacy both relies on and develops. 21st century skills are discussed in greater detail in the next section of this framework. For now, it should be stressed that while contexts (personal, societal, occupational and scientific) influence the development of test items, there is no expectation that items will be deliberately developed to incorporate or address 21st century skills. Instead, the expectation is that by responding to the spirit of the framework and in line with the definition of mathematical literacy, the 21st century skills that have been identified will automatically be incorporated in the items. .

37. The language of the definition and the representation in Figure 1 and Figure 2 quite clearly retain and integrate the notion of mathematical modelling, which has historically been a cornerstone of the PISA framework for mathematics e.g. (OECD, 2004[6]; OECD, 2013[7]). The modelling cycle (formulate, employ, interpret and evaluate) is a central aspect of the PISA conception of mathematically literate students; however, it is often not necessary to engage in every stage of the modelling cycle, especially in the context of an assessment (Galbraith, Henn and Niss, 2007[19]). It is often the case that significant parts of the mathematical modelling cycle have been undertaken by others, and the end user carries out some of the steps of the modelling cycle, but not all of them. For example, in some cases, mathematical representations, such as graphs or equations, are given that can be directly manipulated in order to answer some question or to draw some conclusion. In other cases, students may be using a computer simulation to explore the impact of variable change in a system or environment. For this reason, many PISA items involve only parts of the modelling cycle. In reality, the problem solver may also sometimes oscillate between the processes, returning to revisit earlier decisions and assumptions. Each of the processes may present considerable challenges, and several iterations around the whole cycle may be required.

38. In particular, the verbs ‘formulate’, ‘employ’ and ‘interpret’ point to the three processes in which students as active problem solvers will engage. Formulating situations mathematically involves identifying opportunities to apply and use mathematics – seeing that mathematics can be applied to understand or resolve a particular problem or challenge presented. It includes being able to take a situation as presented and transform it into a form amenable to mathematical treatment, providing mathematical structure and representations, identifying variables and making simplifying assumptions to help solve the problem or meet the challenge. Employing mathematics involves applying mathematical reasoning while using mathematical concepts, procedures, facts and tools to derive a mathematical solution. It includes performing calculations, manipulating algebraic expressions and equations or other mathematical models, analysing information in a mathematical manner from mathematical diagrams and graphs, developing mathematical descriptions and explanations and using mathematical tools to solve problems. Interpreting mathematics involves reflecting upon mathematical solutions or results and interpreting them in the context of a problem or challenge. It includes evaluating mathematical solutions or reasoning in relation to the context of the problem and determining whether the results are reasonable and make sense in the situation.

39. Included for the first time in the PISA 2021 framework is an appreciation of the intersection between mathematical and computational thinking engendering a similar set of perspectives, thought processes and mental models that learners need to succeed in an increasingly technological world. A set of constituent practices recursively positioned under the computational thinking umbrella (namely abstraction, algorithmic thinking, automation, decomposition and generalisation) are also central to both

mathematical reasoning and problem solving process. The nature of computational thinking within mathematics is conceptualised as defining and elaborating mathematical knowledge that can be expressed by programming, allowing students to dynamically model mathematical concepts and relationships. A taxonomy of computational thinking practices geared specifically towards mathematics and science learning entails data practices, modelling and simulation practices, computational problem solving practices, and systems thinking practices (Weintrop et al., 2016[15]). The combination of mathematical thinking (domain-specific) and computational thinking (domain-general) primitives not only becomes essential to effectively support the development of students' conceptual understanding of the mathematical domain, but also to develop their computational thinking concepts and skills, giving learners a more realistic view of how mathematics is practiced in the professional world and better preparing them for pursuing careers in related fields (Basu et al., 2016[20]; Benton et al., 2017[21]; Pei, Weintrop and Wilensky, 2018[14]; Beheshti et al., 2017[22]).

An Explicit Link to a Variety of Contexts for Problems in PISA 2021

40. The reference to 'a variety of real-world contexts' in the definition of mathematical literacy is purposeful and intended as a way to link to the specific contexts that are described and exemplified more fully later in this framework. The specific contexts themselves are not so important, but the four categories selected for use here (personal, occupational, societal and scientific) reflect a wide range of situations in which individuals may meet mathematical opportunities. The definition also acknowledges that mathematical literacy helps individuals to recognise the role that mathematics plays in the world and to make the kinds of well-founded judgments and decisions required of constructive, engaged and reflective citizens.

A Visible Role for Mathematical Tools, including Technology in PISA 2021

41. The definition of mathematical literacy explicitly includes the use of mathematical tools. These tools include a variety of physical and digital equipment, software and calculation devices. Computer-based mathematical tools are in common use in workplaces of the 21st century, and will be increasingly more prevalent as the century progresses both in the workplace and in society generally. The nature of day to day and work-related problems and the demands on individuals to be able to employ mathematical and logical reasoning in situations where computational tools are present has expanded with these new opportunities – creating enhanced expectations for mathematical literacy.

42. Since the 2015 cycle, computer based assessment (CBA) has been the primary mode of testing, although an equivalent paper-based instrument is available for those countries who chose not to test their students by computer. The 2015 and 2018 mathematical literacy assessments were however largely computer neutral.

43. Computer Based Assessment of Mathematics (CBAM) will be the format of the mathematical literacy from 2021. Although the option of a paper based assessment will remain for countries who want to continue in that way, the CBAM will no longer be a computer neutral version of a paper assessment. The opportunities that this transition creates are discussed in greater detail later in the framework.

ORGANISATION OF THE DOMAIN

44. The PISA mathematics framework defines the domain of mathematics for the PISA survey and describes an approach to the assessment of the mathematical literacy of 15-year-olds. That is, PISA assesses the extent to which 15-year-old students can reason mathematically and handle mathematics adeptly when confronted with situations and problems – the majority of which are presented in real-world contexts.

45. For purposes of the assessment, the PISA 2021 definition of mathematical literacy can be analysed in terms of three interrelated aspects:

- Mathematical reasoning and problem solving (which includes the mathematical processes that describe what individuals do to connect the context of the problem with mathematics and thus solve the problem);
- The mathematical content that is targeted for use in the assessment items; and
- The contexts in which the assessment items are located coupled with selected[1] 21st century skills that support and are developed by mathematical literacy.

46. The following sections elaborate these aspects. In highlighting these aspects of the domain, the PISA 2021 mathematics framework helps to ensure that assessment items developed for the survey reflect a range of mathematical reasoning and problem solving, content, and contexts and 21st century skills, so that, considered as a whole, the set of assessment items effectively operationalises what this framework defines as mathematical literacy. Several questions, based on the PISA 2021 definition of mathematical literacy lie behind the organisation of this section of the framework. They are:

- What do individuals engage in when reasoning mathematically and solving contextual mathematical problems?
- What mathematical content knowledge can we expect of individuals – and of 15-year-old students in particular?
- In what context is mathematical literacy able to be both observed and assessed and how do these interact with the identified 21st century skills?

Mathematical Reasoning and Problem Solving Processes

Mathematical reasoning

47. The ability to reason logically and to present arguments in honest and convincing ways is a skill which is becoming increasingly important in today's world. Mathematics

[1] The selected skills were recommended by the OECD Subject Advisory Group (SAG) (*PISA 2021 Mathematics: A Broadened Perspective* [EDU/PISA/GB(2017)17] by finding the union between generic 21st Century skills and related but subject-matter specific skills that are a natural part of the instruction related in the subject matter. The advisory group identified eight 21st Century skills for inclusion in the mathematics curriculum and, as such, in the PISA 2021 assessment framework. These skills are listed in paragraph 124.

is a science about well-defined objects and notions which can be analysed and transformed in different ways using 'mathematical reasoning' to obtain timeless conclusions about which we are certain. In mathematics, students learn that with proper reasoning and assumptions they can arrive at results which they can trust to be true. Further, those conclusions are logical and objective, and hence impartial, without any need for validation by an external authority.

48. This kind of reasoning is useful far beyond mathematics, but it can be learned and practiced most effectively within mathematics, because it has the advantage of a fully-defined context, which creates a comfortable training environment and under the assumed axioms students experience objective truth.

49. Mathematical reasoning has two aspects, both important in today's world. One is deduction from clear assumptions, which is a characteristic feature of 'pure' mathematics. The usefulness of this ability has been stressed above.

50. The second important dimension is statistical and probabilistic reasoning. At the logical level, there is nowadays constant confusion in the minds of individuals between the possible and the probable, leading many to fall prey to conspiracy theories or fake news. From a technical perspective, today's world is increasingly complex and its multiple dimensions are represented by terabytes of data. Making sense of these data is one of the biggest challenges that humanity will face in the future. Our students should be familiarised with the nature of such data and making informed decisions in the context of variation and uncertainty.

51. The power of mathematics, from its very beginnings, lies in the ability of reducing complex contexts to fundamental principles. Good mathematics education should develop the capability of seeking these 'prime principles' in well designed, yet quite complicated contexts.

52. Mathematical reasoning, supported by a small number of "big mathematical ideas" that undergird the specific content, skills, and algorithms of school mathematics, is the core of mathematical literacy. There are at least six "big ideas" that provide structure and support to mathematical reasoning. These "big ideas" are the following:

- quantity, number systems and their algebraic properties;
- mathematics as a system based on abstraction and symbolic representation;
- mathematical structure and its regularities;
- functional relationships between quantities;
- mathematical modelling as a lens onto the real world (e.g. those arising in the physical, biological, social, economic, and behavioural sciences); and
- variance as the heart of statistics.

The description of each of the "big ideas" that follows provides an overview of the ideas. While the descriptions may appear abstract, the intention is not for them to be treated in an abstract way in the PISA assessment. The message that the descriptions should convey is how these ideas surface throughout school mathematics and how, by reinforcing their occurrence in teaching we support students to realise how they can be applied in new and different contexts.

Quantity, number systems and their algebraic properties

53. The basic notion of quantity may be the most pervasive and essential mathematical aspect of engaging with, and functioning in, the world (OECD, 2017,

p. 18[23]). At the most basic level it deals with the useful ability to compare cardinalities of sets of objects. The ability to count usually involves rather small sets – in most languages, only a small subset of numbers have names. When we assess larger sets, we engage in more complex operations of estimating, rounding and applying orders of magnitude. Counting is very closely related to another fundamental operation of classifying things, where the ordinal aspect of numbers emerges. Quantification of attributes of objects, relationships, situations and entities in the world is one of the most basic ways of conceptualising the surrounding world (OECD, 2017[23]).

54. Beyond counting, number is fundamental to measurement, which some would argue is an essential practice in using mathematics to solve problems about our world. As the Scots-Irish mathematician Lord Kelvin once claimed in a lecture delivered at the London-based Institution of Civil Engineers on 3 May 1883: “When you can measure what you are speaking about and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind.”

55. This “big idea” includes the basic concept of number, nested number systems (e.g., whole numbers to integers to rationals to reals), the arithmetic of numbers, and the algebraic properties that the systems enjoy. In particular, it is useful to understand how progressively more expansive systems of numbers enable the solution of progressively more complicated equations. This lays the foundation for enabling students to see the structures of mathematics in the real world in the later part of their lives.

56. To use quantification efficiently, one has to be able to apply not just numbers, but the number systems. Numbers themselves are of limited relevance; what makes them into a powerful tool are the operations that we can perform with them. As such, a good understanding of the operations of numbers is the foundation of mathematical reasoning.

57. It is also important to understand matters of representation (as symbols involving numerals, as points on a number line, as geometric quantities, and by special symbols such as *pi* and *e*[2]) and how to move between them; the ways in which these representations are affected by number systems; the ways in which algebraic properties of these systems are relevant and matter for operating within the systems; and the significance of the additive and multiplicative identities, associativity, commutativity, and the distributive property of multiplication over addition. Algebraic principles undergird the place value system, allowing for economical expression of numbers and efficient approaches to operations on them. They are also central to number-line based operations with numbers, including work with additive inverses that are central to addition and subtraction of first integers and then reals.

58. The centrality of number as a key concept in all the other mathematical areas under consideration here and to mathematical reasoning itself, is undeniable. Students’ grasp of the algebraic principles and properties first experienced through work with numbers is fundamental to their understanding of the concepts of secondary school algebra, along with their ability to become fluent in the manipulations of algebraic expressions necessary for solving equations, setting up models, graphing functions, and programing and making spreadsheet formulas. And in today’s data-intensive world,

[2] Note: This is a general statement about the number systems and their algebraic properties and should not be taken to mean that students doing PISA are expected to be familiar with *e* or complex equations.

facility with interpretation of patterns of numbers, comparison of patterns, and other numerical skills are evolving in importance.

Mathematics as a system based on abstraction and symbolic representation

59. The fundamental ideas of mathematics have arisen from human experience in the world and the need to provide coherence, order, and predictability to that experience. Many mathematical objects model reality, or at least reflect aspects of reality in some way. However, the essence of abstraction in mathematics is that it is a self-contained system, and mathematics objects derive their meaning from within that system. Abstraction involves deliberately and selectively attending to structural similarities between mathematical objects, and constructing relationships between those objects based on these similarities. In school mathematics, abstraction forms relationships between concrete objects, symbolic representations and operations including algorithms and mental models. This ability also plays a role in working with computational devices. The ability to create, manipulate, and draw meaning in working with abstractions in technological contexts in an important computational thinking skill.

60. For example, children begin to develop the concept of "circle" by experiencing specific objects that lead them to an informal understanding of circles as being "perfectly round". They might draw circles to represent these objects, noticing similarities between the drawings to generalise about "roundness" even though the circles are of different sizes. "Circle" becomes an abstract mathematics object only when it is defined as the locus of points equidistant from a fixed point in a two-dimensional plane.

61. Students use representations – whether symbolic, graphical, numerical, or geometric – to organise and communicate their mathematical thinking. Representations enable us to present mathematical ideas in a succinct way which, in turn, lead to efficient algorithms. Representations are also a core element of mathematical modelling, allowing students to abstract a simplified or idealised formulation of a real world problem. Such structures are also important for interpreting and defining the behaviour of computational devices.

The structures of mathematics and their regularities

62. When elementary students see: $5 + (3 + 8)$ some see a string of symbols indicating a computation to be performed in a certain order according to the rules of order of operations; others see a number added to the sum of two other numbers. The latter group are seeing structure; and because of that they don't need to be told about the order of operation, because if you want to add a number to a sum you first have to compute the sum.

63. Seeing structure continues to be important as students move to higher grades. A student who sees $f(x) = 5 + (x - 3)^2$ as saying that $f(x)$ is the sum of 5 and a square which is zero when $x = 3$ understands that the minimum of f is 5. This lays the foundation for functional thinking discussed in the next section.

64. Structure is intimately related to symbolic representation. The use of symbols is powerful, but only if they retain meaning for the symboliser, rather than becoming meaningless objects to be rearranged on a page. Seeing structure is a way of finding and remembering the meaning of an abstract representation. Such structures are also important for interpreting and defining the behaviour of computational devices. Being able to see structure is an important conceptual aid to purely procedural knowledge.

65. What is the relationship between mathematical structure as a plausible generalisation of a mathematical concept and reasoning? As the examples above illustrate, seeing structure in abstract mathematical objects is a way of replacing parsing rules, which can be performed by a computer, with conceptual images of those objects that make their properties clear. An object held in the mind in such a way is subject to reasoning at a level that is higher than simple symbolic manipulation.

66. A robust sense of mathematical structure also supports modelling. When the objects under study are not abstract mathematical objects, but rather objects from the real world to be modelled by mathematics, then mathematical structure can guide the modelling. Students can also impose structure on non-mathematical objects in order to make them subject to mathematical analysis. An irregular shape can be approximated by simpler shapes whose area is known. A geometric pattern can be understood by hypothesising translational, rotational, or reflectional transformations and symmetry and abstractly extending the pattern into all of space. Statistical analysis is often a matter of imposing a structure on a set of data, for example by assuming it comes from a normal distribution.

Functional relationships between quantities

67. Students in elementary school encounter problems where they must find specific quantities. For example, how fast do you have to drive to get from Tucson to Phoenix, a distance of 180 km, in 1 hour and 40 minutes? Such problems have a specific answer: to drive 180 km in 1 hour and 40 minutes you must drive at 108 km per hour.

68. At some point students start to consider situations where quantities are variable, that is, where they can take on a range of values. For example, what is the relation between the distance driven, d, in kilometres, and time spent driving, t, in hours, if you drive at a constant speed of 108 *km* per hour? Such questions introduce functional relationships. In this case the relationship, expressed by the equation $d = 108t$, is a proportional relationship, the fundamental example and perhaps the most important for general knowledge.

69. Relationships between quantities can be expressed with equations, graphs, tables, or verbal descriptions. An important step in learning is to extract from these the notion of a function itself, as an abstract object of which these are representations. The essential elements of the concept are a domain, from which inputs are selected, a codomain, in which outputs lie, and a process for producing outputs from inputs.

70. Explicitly noting the domain and codomain allows for many different topics to be brought under the function concept. A parametric curve is a function whose domain is a subset of the real numbers and whose codomain is two- or three-dimensional space. Arithmetic operations can be viewed as functions whose domain is the set of ordered pairs of numbers. Geometric formulas for circumference, area, surface area, and volume can be viewed as functions whose domain is the set of geometric objects. Geometric transformations, such as translations, rotations, reflections, and dilations, can be viewed as functions from space to itself.

71. The more formal definition of a function as a set of ordered pairs is both problematic and useful in school mathematics. It is problematic because it removes the dynamic aspect of students' conceptualisation of function: function as process, or mapping, or coordination of two varying quantities. These conceptualisations are useful in many common uses of functions in science, computer science, society, and everyday

life. On the other hand, the ordered-pair definition emphasises the invariance of the function as an object in its own right, independent of different methods of computing its outputs from its inputs. Thus different forms for the expression of a quadratic function, say $f\ (x) = (x - 1)(x - 3)$ and $f(x) = (x - 2)^2 - 1$, throw light on different properties of the same object: the one shows its zeros, the other its minimum value.

72. The two views of function – the naïve view as a process and the more abstract view as an object – can be reconciled in the graph of the function. As a set of ordered pairs it is a manifestation of the object. But reading a graph, coordinating the values on the axes, also has a dynamic or process aspect. And the graph of a function is an important tool for exploring the notion of a rate of change. The graph provides a visual tool for understanding a function as a relationship between co-varying quantities.

Mathematical modelling as a lens onto the real world

73. Models represent a conceptualisation of phenomena. Models are simplifications of reality that fore- ground certain features of a phenomenon while approximating or ignoring other features. As such, ‘‘all models are wrong, but some are useful’’ (Box and Draper, 1987, p. 424[24]). The usefulness of a model comes from its explanatory and/or predictive power (Weintrop et al., 2016[15]). Models are, in that sense, abstractions of reality. A model may present a conceptualisation that is understood to be an approximation or working hypothesis concerning the object phenomenon or it may be an intentional simplification. Mathematical models are formulated in mathematical language and use a wide variety of mathematical tools and results (e.g., from arithmetic, algebra, geometry, etc.). As such, they are used as ways of precisely defining the conceptualisation or theory of a phenomenon, for analysing and evaluating data (does the model fit the data?), and for making predictions. Models can be operated – that is, made to run over time or with varying inputs, thus producing a simulation. When this is done, it is possible to make predictions, study consequences, and evaluate the adequacy and accuracy of the models. Throughout the modelling process cognisance needs to be taken of the real world parameters that impact on the model and the solutions developed using the model.

74. Computer-based (or computational) models provide the ability to test hypothesis, generate data, introduce randomness and so on. Mathematical literacy includes the ability to understand, evaluate and draw meaning from computational models.

Variance as the heart of statistics

75. Living things as well as non-living things vary with respect to many characteristics. However, as a result of that typically large variation, it is difficult to make generalisations in such a world without characterising in some way to what extent that generalisation holds. In statistics accounting for variability is one, if not the central, defining element around which the discipline is based. In today’s world people often deal with these types of situations by merely ignoring the variation and as a result suggesting sweeping generalisations which are often misleading, if not wrong, and as a result very dangerous. Bias in the social science sense is usually created by not accounting for the variability in the trait under discussion.

76. Statistics is essentially about accounting for or modelling variability as measured by the variance or in the case of multiple variables the covariance matrix. This provides a probabilistic environment in which to understand various phenomena as well as to make

critical decisions. Statistics is in many ways a search for patterns in a highly variable context: trying to find the signal defining "truth" in the midst of a great deal of random noise. "Truth" is set in quotes as it is not the Platonic truth that mathematics can deliver but an estimate of truth set in a probabilistic context, accompanied by an estimate of the error contained in the process. Ultimately, the decision maker is left with the dilemma of never knowing for certain what the truth is. The estimate that has been developed is, at best and range of possible values – the better the process the narrower the range of possible values, although a range cannot be avoided.

Problem solving

77. The definition of mathematical literacy refers to an individual's capacity to formulate, employ, and interpret (and evaluate) mathematics. These three words, formulate, employ and interpret, provide a useful and meaningful structure for organising the mathematical processes that describe what individuals do to connect the context of a problem with the mathematics and solve the problem. Items in the 2021 PISA mathematics test will be assigned to either mathematical reasoning or one of three mathematical processes:

- Formulating situations mathematically;
- Employing mathematical concepts, facts, procedures and reasoning; and
- Interpreting, applying and evaluating mathematical outcomes.

78. It is important for both policy makers and those engaged more closely in the day-to-day education of students to know how effectively students are able to engage in each of these elements of the problem solving model/cycle. Formulating indicates how effectively students are able to recognise and identify opportunities to use mathematics in problem situations and then provide the necessary mathematical structure needed to formulate that contextualised problem in a mathematical form. Employing refers to how well students are able to perform computations and manipulations and apply the concepts and facts that they know to arrive at a mathematical solution to a problem formulated mathematically. *Interpreting* (and *evaluating*) relates to how effectively students are able to reflect upon mathematical solutions or conclusions, interpret them in the context of the real-world problem and determine whether the result(s) or conclusion(s) are reasonable and/or useful. Students' facility at applying mathematics to problems and situations is dependent on skills inherent in all three of these stages, and an understanding of students' effectiveness in each category can help inform both policy-level discussions and decisions being made closer to the classroom level.

79. Moreover, introducing students to mathematical problem solving processes through computational thinking tools and practices encourage students to practice prediction, reflection and debugging skills (Brennan and Resnick, 2012[25]). The use of new blocks-based programming languages, designed to make complex concepts more accessible using visual cues (e.g. colour and shape), helps students to overcome difficulties in syntax enabling more pupils to gain a deeper understanding of mathematical ideas through the programming activities (Benton et al., 2017[21]; Resnick et al., 2009[18]; Weintrop and Wilensky, 2015[26]).

Formulating Situations Mathematically

80. The word *formulate* in the mathematical literacy definition refers to individuals being able to recognise and identify opportunities to use mathematics and then provide mathematical structure to a problem presented in some contextualised form. In

the process of formulating situations mathematically, individuals determine where they can extract the essential mathematics to analyse, set up and solve the problem. They translate from a real-world setting to the domain of mathematics and provide the real-world problem with mathematical structure, representations and specificity. They reason about and make sense of constraints and assumptions in the problem. Specifically, this process of formulating situations mathematically includes activities such as the following:

- selecting an appropriate model from a list;[3]
- identifying the mathematical aspects of a problem situated in a real-world context and identifying the significant variables;
- recognising mathematical structure (including regularities, relationships, and patterns) in problems or situations;
- simplifying a situation or problem in order to make it amenable to mathematical analysis (for example by decomposing);
- identifying constraints and assumptions behind any mathematical modelling and simplifications gleaned from the context;
- representing a situation mathematically, using appropriate variables, symbols, diagrams, and standard models;
- representing a problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions;
- understanding and explaining the relationships between the context-specific language of a problem and the symbolic and formal language needed to represent it mathematically;
- translating a problem into mathematical language or a representation;
- recognising aspects of a problem that correspond with known problems or mathematical concepts, facts or procedures;
- choosing among an array of and employing the most effective computing tool to portray a mathematical relationship inherent in a contextualised problem; and
- creating an ordered series of (step-by-step) instructions for solving problems.

Employing Mathematical Concepts, Facts, Procedures and Reasoning

81. The word *employ* in the mathematical literacy definition refers to individuals being able to apply mathematical concepts, facts, procedures, and reasoning to solve mathematically-formulated problems to obtain mathematical conclusions. In the process of employing mathematical concepts, facts, procedures and reasoning to solve problems, individuals perform the mathematical procedures needed to derive results and find a mathematical solution (e.g. performing arithmetic computations, solving equations, making logical deductions from mathematical assumptions, performing symbolic manipulations, extracting mathematical information from tables and graphs, representing and manipulating shapes in space, and analysing data). They work on a model of the problem situation, establish regularities, identify connections between mathematical entities, and create mathematical arguments. Specifically, this process of employing mathematical concepts, facts, procedures and reasoning includes activities such as:

- performing a simple calculation;[4] **

[3] This activity is included in the list to foreground the need for the test items developers to include items that are accessible to students at the lower end of the performance scale.

- drawing a simple conclusion; **
- selecting an appropriate strategy from a list; **
- devising and implementing strategies for finding mathematical solutions;
- using mathematical tools, including technology, to help find exact or approximate solutions;
- applying mathematical facts, rules, algorithms, and structures when finding solutions;
- manipulating numbers, graphical and statistical data and information, algebraic expressions and equations, and geometric representations;
- making mathematical diagrams, graphs, simulations, and constructions and extracting mathematical information from them;
- using and switching between different representations in the process of finding solutions;
- making generalisations based on the results of applying mathematical procedures to find solutions;
- reflecting on mathematical arguments and explaining and justifying mathematical results; and
- evaluating the significance of observed (or proposed) patterns and regularities in data.

Interpreting, Applying and Evaluating Mathematical Outcomes

82. The word *interpret* (and *evaluate*) used in the mathematical literacy definition focuses on the ability of individuals to reflect upon mathematical solutions, results or conclusions and interpret them in the context of the real-life problem that initiated the process. This involves translating mathematical solutions or reasoning back into the context of the problem and determining whether the results are reasonable and make sense in the context of the problem. *Interpreting, applying and evaluating mathematical outcomes* encompasses both the 'interpret' and 'evaluate' elements of the mathematical modelling cycle. Individuals engaged in this process may be called upon to construct and communicate explanations and arguments in the context of the problem, reflecting on both the modelling process and its results. Specifically, this process of interpreting, applying and evaluating mathematical outcomes includes activities such as:

- interpreting information presented in graphical form and/or diagrams;[5] **
- evaluating a mathematical outcome in terms of the context; **
- interpreting a mathematical result back into the real-world context;
- evaluating the reasonableness of a mathematical solution in the context of a real-world problem;
- understanding how the real world impacts the outcomes and calculations of a mathematical procedure or model in order to make contextual judgments about how the results should be adjusted or applied;

[4] These activities (**) are included in the list to foreground the need for the test items developers to include items that are accessible to students at the lower end of the performance scale.

[5] These activities (**) are included in the list to foreground the need for the test items developers to include items that are accessible to students at the lower end of the performance scale

- explaining why a mathematical result or conclusion does, or does not, make sense given the context of a problem;
- understanding the extent and limits of mathematical concepts and mathematical solutions;
- critiquing and identifying the limits of the model used to solve a problem; and
- using mathematical thinking and computational thinking to make predictions, to provide evidence for arguments, to test and compare proposed solutions.

Mathematical Content Knowledge

83. An understanding of mathematical content – and the ability to apply that knowledge to solving meaningful contextualised problems – is important for citizens in the modern world. That is, to reason mathematically and to solve problems and interpret situations in personal, occupational, societal and scientific contexts, there is a need to draw upon certain mathematical knowledge and understanding.

84. Since the goal of PISA is to assess mathematical literacy, an organisational structure for mathematical content knowledge is proposed that is based on mathematical phenomena that underlie broad classes of problems. Such an organisation for content is not new, as exemplified by two well-known publications: On the Shoulders of Giants: New Approaches to Numeracy (Steen, 1990[27]) and Mathematics: The Science of Patterns (Devlin, 1994[28]).

85. The following content categories (previously used in 2012) are again used in PISA 2021 to reflect both the mathematical phenomena that underlie broad classes of problems, the general structure of mathematics, and the major strands of typical school curricula. These four categories characterise the range of mathematical content that is central to the discipline and illustrate the broad areas of content used in the test items for PISA 2021 (which will include PISA-D items to increase opportunities at the lower end of the performance spectrum):

- change and relationships
- space and shape
- quantity
- uncertainty and data

86. With these four categories, the mathematical domain can be organised in a way that ensures a spread of items across the domain and focuses on important mathematical phenomena, while at the same time, avoiding too granular a classification that would prevent the analysis of rich and challenging mathematical problems based on real situations.

87. While categorisation by content category is important for item development, selection and reporting of the assessment results, it is important to note that some items could potentially be classified in more than one content category. For example, a released PISA item called Pizzas involves determining which of two round pizzas, with different diameters and different costs but the same thickness, is the better value (see Appendix B to view this item and an analysis of its attributes). This item could be thought of in different ways. On the one hand it could be analysed using measurement and quantification (value for money, proportional reasoning and arithmetic calculations). On the other hand, it could the thought of as a change and relationships situation (in terms of relationships among the variables and how relevant properties change from the smaller

pizza to the larger one). This item was ultimately categorised as a change and relationships item since the key to the problem lies in students being able to relate the change in areas of the two pizzas (given a change in diameter) and a corresponding variation price. Clearly, a different item involving circle area might be classified as a space and shape item.

88. National school mathematics curricula are typically organised around content strands (most commonly: numbers, algebra, functions, geometry, and data handling) and detailed topic lists help to define clear expectations. These curricula are designed to equip students with knowledge and skills that address these same underlying mathematical phenomena that organise the PISA content. The outcome is that the range of content arising from organising it in the way that PISA does is closely aligned with the content that is typically found in national mathematics curricula. This framework lists some a range of content topics appropriate for assessing the mathematical literacy of 15-year-old students, based on analyses of national standards from eleven countries.

89. The broad mathematical content categories and the more specific content topics appropriate for 15-year-old students described in this section reflect the level and breadth of content that is eligible for inclusion in the PISA 2021 assessment. Descriptions of each content category and the relevance of each to reasoning and solving meaningful problems are provided, followed by more specific definitions of the kinds of content that are appropriate for inclusion in an assessment of mathematical literacy of 15-year-old students and out of school youth.

90. Four topics have been identified for special emphasis in the PISA 2021 assessment. These topics are not new to the mathematics content categories. Instead, these are topics within the existing content categories that deserve special emphasis. In the work of Mahajan et al. ("PISA Mathematics 2021", (2016[29])) the four topics are presented not only as commonly encountered situations in adult life in general, but as the types of mathematics needed in the emerging new areas of the economy such as high-tech manufacturing etc. The four are: growth phenomena; geometric approximations; computer simulations; and conditional decision making. Each topic is discussed with the discussion of the corresponding content category as follows:

- Growth phenomena (change and relationships)
- Geometric approximation (space and shape)
- Computer simulations (quantity)
- Conditional decision making (uncertainty and data)

Change and Relationships

91. The natural and designed worlds display a multitude of temporary and permanent relationships among objects and circumstances, where changes occur within systems of interrelated objects or in circumstances where the elements influence one another. In many cases these changes occur over time, and in other cases changes in one object or quantity are related to changes in another. Some of these situations involve discrete change; others change continuously. Some relationships are of a permanent, or invariant, nature. Being more literate about change and relationships involves understanding fundamental types of change and recognising when they occur in order to use suitable mathematical models to describe and predict change. Mathematically this means modelling the change and the relationships with appropriate functions and equations, as

well as creating, interpreting and translating among symbolic and graphical representations of relationships.

92. Change and relationships is evident in such diverse settings as growth of organisms, music, seasonal change and cycles, weather patterns, employment levels and economic conditions. Aspects of the traditional mathematical content of functions and algebra, including algebraic expressions, equations and inequalities, tabular and graphical representations, are central in describing, modelling and interpreting change phenomena. Computational tools provide a means to visualize and interact with change and relationships. Recognizing how and when a computational device can augment and complement mathematical concepts is an important computational thinking skill.

93. Representations of data and relationships described using statistics are also used to portray and interpret change and relationships, and a firm grounding in the basics of number and units is also essential to defining and interpreting change and relationships. Some interesting relationships arise from geometric measurement, such as the way that changes in perimeter of a family of shapes might relate to changes in area, or the relationships among lengths of the sides of triangles.

94. Growth phenomena: Understanding the dangers of flu pandemics and bacterial outbreaks, as well as the threat of climate change, demand that people think not only in terms of linear relationships but recognise that such phenomena need non-linear (often exponential but also other) models. Linear relationships are common and are easy to recognise and understand but to assume linearity can be dangerous. A good example of linearity and one probably used by everyone is estimating the distance travelled in various amounts of time while traveling at a given speed. Such an application provides a reasonable estimate as long as the speed stays relatively constant. But with flu epidemics, for example, such a linear approach would grossly underestimate the number of people sick in 5 days after the initial outbreak. Here is where a basic understanding of non-linear (including quadratic and exponential) growth and how rapidly infections can spread given that the rate of change increases from day to day is critical. The spread of the Zika infection is an important example of exponential growth; recognising it as such helped medical personnel to understand the inherent threat and the need for fast action.

95. Identifying growth phenomena as a focal point of the change and relationships content category is not to signal that there is an expectation that participating students should have studied the exponential function and certainly the items will not require knowledge of the exponential function. Instead, the expectation is that there will be items that expect students to (a) recognise that not all growth is linear, (b) that non-linear growth has particular and profound implications on how we understand certain situations, and (c) appreciate the intuitive meaning of “exponential growth” as an extremely rapid rate of growth, for example in the earthquake scale, every increase by 1 unit on the Richter scale does not mean a proportional increase in its effect, but rather by 10, 100, and 1000 times etc.

Space and Shape

96. Space and shape encompasses a wide range of phenomena that are encountered everywhere in our visual and physical world: patterns, properties of objects, positions and orientations, representations of objects, decoding and encoding of visual information, navigation and dynamic interaction with real shapes as well as with representations,

movement, displacement, and the ability to anticipate actions in space. Geometry serves as an essential foundation for space and shape, but the category extends beyond traditional geometry in content, meaning and method, drawing on elements of other mathematical areas such as spatial visualisation, measurement and algebra. For instance, shapes can change and a point can move along a locus, thus requiring function concepts. Measurement formulas are central in this area. The recognition, manipulation and interpretation of shapes in settings that call for tools ranging from dynamic geometry software to Global Positioning Systems (GPS), and to machine learning software are included in this content category.

97. PISA assumes that the understanding of a set of core concepts and skills is important to mathematical literacy relative to space and shape. Mathematical literacy in the area of space and shape involves a range of activities such as understanding perspective (for example in paintings), creating and reading maps, transforming shapes with and without technology, interpreting views of three-dimensional scenes from various perspectives and constructing representations of shapes.

98. Geometric approximations: Today's world is full of shapes that do not follow typical patterns of evenness or symmetry. Because simple formulas do not deal with irregularity, it has become more difficult to understand what we see and find the area or volume of the resulting structures. For example, finding the needed amount of carpeting in a building in which the apartments have acute angles together with narrow curves demands a different approach than would be the case with a typically rectangular room.

99. Identifying geometric approximations as a focal point of the space and shape content category signals the need for students to be able use their understanding of traditional space and shape phenomena in a range of typical situations.

Quantity

100. The notion of quantity may be the most pervasive and essential mathematical aspect of engaging with, and functioning in, our world. It incorporates the quantification of attributes of objects, relationships, situations and entities in the world, understanding various representations of those quantifications and judging interpretations and arguments based on quantity. To engage with the quantification of the world involves understanding measurements, counts, magnitudes, units, indicators, relative size and numerical trends and patterns. Aspects of quantitative reasoning – such as number sense, multiple representations of numbers, elegance in computation, mental calculation, estimation and assessment of reasonableness of results – are the essence of mathematical literacy relative to quantity.

101. Quantification is a primary method for describing and measuring a vast set of attributes of aspects of the world. It allows for the modelling of situations, for the examination of change and relationships, for the description and manipulation of space and shape, for organising and interpreting data and for the measurement and assessment of uncertainty. Thus mathematical literacy in the area of quantity applies knowledge of number and number operations in a wide variety of settings.

102. Computer simulations: Both in mathematics and statistics there are problems that are not so easily addressed because the required mathematics are complex or involve a large number of factors all operating in the same system or because of ethical issues relating to the impact on living beings or their environment. Increasingly in today's world such problems are being approached using computer simulations driven by algorithms.

A good example is the use of such simulations towards helping individuals plan their retirement so as to have enough money on which to live and accomplish their goals. The number of factors to consider is very large. They include income, age of retirement, expected expenses, investment earnings, stock market values, the expected age at death, the role of children in providing support and so on, the values of which all require a set of assumptions to be made by the individual or the computer program. Changing any of those assumptions individually or collectively provides different results which can then be aggregated statistically to provide an overall estimate of how achievable retirement is, given the goals. Users of such a simulation program need to understand at some level how this is done so as to interpret the results as to the implied impact of their assumptions.

103. Identifying computer simulations as a focal point of the quantity content category signals that in the context the Computer Based Assessment of Mathematics (CBAM) of PISA being used from 2021, there are a broad category of complex problems including budgeting and planning that students can analyse in terms of the variables of the problem using computer simulations provided as part of the test item.

Uncertainty and Data

104. In science, technology and everyday life, uncertainty is a given. Uncertainty is therefore a phenomenon at the heart of the mathematical analysis of many problem situations, and the theory of probability and statistics as well as techniques of data representation and description have been established to deal with it. The uncertainty and data content category includes recognising the place of variation in processes, having a sense of the quantification of that variation, acknowledging uncertainty and error in measurement and knowing about chance. It also includes forming, interpreting and evaluating conclusions drawn in situations where uncertainty is central. The presentation and interpretation of data are key concepts in this category (Moore, 1997[30]).

105. There is uncertainty in scientific predictions, poll results, weather forecasts and economic models. There is variation in manufacturing processes, test scores and survey findings, and chance is fundamental to many recreational activities enjoyed by individuals. The traditional curricular areas of probability and statistics provide formal means of describing, modelling and interpreting a certain class of uncertainty phenomena, and for making inferences. In addition, knowledge of number and of aspects of algebra such as graphs and symbolic representation contribute to facility in engaging in problem solving in this content category.

106. Conditional decision making: Going to the doctor often ends up requiring a decision about what to do next. Do I take the medicine? Given my age, the doctor says that I should live another 20 years. The question arises – should I take a low dosage aspirin every day? This is a decision for which certainty is not possible. However, the decision making can be improved by asking the doctor how long I should live and not have a heart attack if I have a particular blood pressure, heart rate, cholesterol-level and body fat ratio. This is a question answerable best in terms of conditional probability. Such a probability or even an estimate of that probability takes into account the additional information such as the patient's current heath as indicated by the blood and other tests. If there is a relationship between blood pressure and cholesterol levels and the other test results in terms of the chances of having a future heart attack, the predicated conditional mean might for example change the life expectancy estimate of 20 years to only 5 years,

impacting decisively on the patient's view of whether to take a daily aspirin. Essentially the conditional mean provides a non-technical approach to regression analysis.

107. Variables which are dichotomous or polytomous in nature must be analysed differently than continuous variables such as blood pressure and age. Questions such as do people with blue eyes also tend to have blonde hair, or how many ways can 20 people be organised into the 11 positions on soccer team are examples of situations where knowing the basic rules of combinatorics would help in making decisions. In the case of the cross-classification of categorical variables, such as blue eyes or not vs. blonde hair or not, understanding the different types of percentages that can be computed and what each of them means is critical to understanding the phenotypic genetics. Combinatorics (into how many different orders can 5 objects be placed? Into how many combinations can 16 objects be placed, taking 2 at a time?) also help to understand commonly occurring situations. Game theory, a more formal approach, can be applied to decision making between different categorical options such as winning vs. losing, using many of these same approaches together with conditional probability.

108. Identifying conditional decision making as a focal point of the uncertainty and data content category signals that students should be expected to appreciate how the assumptions made in setting up a model impact on the conclusions that can be drawn and that different assumptions/relationships may well result in different conclusions. In computational models, conditional logic is made explicit in the instructions defined for the machine. These computational expressions can be useful in representing conditional decision points in a larger process.

Content Topics for Guiding the Assessment of Mathematical Literacy of 15-year-old Students

109. The specific topics listed below reflect commonalities found in the curricular expectations set by a range of countries. The standards examined to identify these content topics are viewed as evidence not only of what is taught in mathematics classrooms in these countries but also as indicators of what countries view as important knowledge and skills for preparing students of this age to become constructive, engaged and reflective 21st century citizens

110. To effectively understand and solve contextualised problems involving change and relationships; space and shape; quantity; and uncertainty and data requires drawing upon a variety of mathematical concepts, procedures, facts, and tools at an appropriate level of depth and sophistication. As an assessment of mathematical literacy, PISA strives to assess the levels and types of mathematics that are appropriate for 15-year-old students on a trajectory to become constructive, engaged and reflective 21st century citizens able to make well-founded judgments and decisions. It is also the case that PISA, while not designed or intended to be a curriculum-driven assessment, strives to reflect the mathematics that students have likely had the opportunity to learn by the time they are 15 years old.

111. In the development of the PISA 2012 mathematical literacy framework, with an eye toward developing an assessment that is both forward-thinking yet reflective of the mathematics that 15-year-old students have likely had the opportunity to learn, analyses were conducted of a sample of desired learning outcomes from eleven countries to determine both what is being taught to students in classrooms around the world and what countries deem realistic and important preparation for students as they approach entry into the workplace or admission into a higher education institution. Based on

commonalities identified in these analyses, coupled with the judgment of mathematics experts, content deemed appropriate for inclusion in the assessment of mathematical literacy of 15-year-old students on PISA 2012, and continued for PISA 2021, is described below.

112. For PISA 2021 four additional focus topics have been added to the list. The resulting lists is intended to be illustrative of the content topics included in PISA 2021 and not an exhaustive listing:

- *Growth phenomena*: Different types of growth: linear, non-linear, quadratic and exponential (the growth of a system in which the amount being added is proportional to the amount already present)
- *Geometric approximation*: Approximating the attributes and properties of irregular or unfamiliar shapes and objects by breaking these shapes and objects up into more familiar shapes and objects for which there are formulae and tools.
- *Computer simulations*: Exploring situations (that may include budgeting, planning, population distribution, disease spread, experimental probability, reaction time modelling etc.) in terms of the variables and the impact that these have on the outcome.
- *Conditional decision making*: Using conditional probability and basic principles of combinatorics to interpret situations and make predictions.
- *Functions*: The concept of function, emphasising but not limited to linear functions, their properties, and a variety of descriptions and representations of them. Commonly used representations are verbal, symbolic, tabular and graphical.
- *Algebraic expressions*: Verbal interpretation of and manipulation with algebraic expressions, involving numbers, symbols, arithmetic operations, powers and simple roots.
- *Equations and inequalities*: Linear and related equations and inequalities, simple second-degree equations, and analytic and non-analytic solution methods.
- *Co-ordinate systems*: Representation and description of data, position and relationships.
- *Relationships within and among geometrical objects in two and three dimensions*: Static relationships such as algebraic connections among elements of figures (e.g. the Pythagorean theorem as defining the relationship between the lengths of the sides of a right triangle), relative position, similarity and congruence, and dynamic relationships involving transformation and motion of objects, as well as correspondences between two- and three-dimensional objects.
- *Measurement*: Quantification of features of and among shapes and objects, such as angle measures, distance, length, perimeter, circumference, area and volume.
- *Numbers and units*: Concepts, representations of numbers and number systems (including converting between number systems), including properties of integer and rational numbers, relevant aspects of irrational numbers, as well as quantities and units referring to phenomena such as time, money, weight, temperature, distance, area and volume, and derived quantities and their numerical description.
- *Arithmetic operations*: The nature and properties of these operations and related notational conventions.
- *Percents, ratios and proportions*: Numerical description of relative magnitude and the application of proportions and proportional reasoning to solve problems.
- *Counting principles*: Simple combinations and permutations.

- *Estimation*: Purpose-driven approximation of quantities and numerical expressions, including significant digits and rounding.
- *Data collection, representation and interpretation*: Nature, genesis and collection of various types of data, and the different ways to analyse, represent and interpret them.
- *Data variability and its description*: Concepts such as variability, distribution and central tendency of data sets, and ways to describe and interpret these in quantitative terms.
- *Samples and sampling*: Concepts of sampling and sampling from data populations, including simple inferences based on properties of samples.
- *Chance and probability*: Notion of random events, random variation and its representation, chance and frequency of events, and basic aspects of the concept of probability.

Contexts for the assessment items and selected 21st century skills

113. The definition of mathematical literacy introduces two important considerations for the PISA assessment items. First, the definition makes it clear that mathematical literacy takes place in *real-world contexts*. Second, mathematical literacy *assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective 21st century citizens*. In this section we discuss how both real-world contexts and 21st century skills impact on item development.

Contexts

114. An important aspect of mathematical literacy is that mathematics is used to solve a problem set in a context. The context is the aspect of an individual's world in which the problems are placed. The choice of appropriate mathematical strategies and representations is often dependent on the context in which a problem arises, and by implication there is the need to utilize knowledge of the real world context in developing the model. Being able to work within a context is widely appreciated to place additional demands on the problem solver (see Watson and Callingham, (2003[31]), for findings about statistics). For PISA, it is important that a wide variety of contexts are used. This offers the possibility of connecting with the broadest possible range of individual interests and with the range of situations in which individuals operate in the 21st century.

115. In light of the number of countries participating in PISA 2021 and with that an increasing range of participants from low- and middle-income countries as well as the possibility of out-of-school 15-year olds, it is important that item developers take great care to ensure that the contexts used for items are accessible to a very broad range of participants. In this regard it is also important that the reading load of the items remains modest so that the items continue to assess mathematical literacy.

116. For purposes of the PISA 2021 mathematics framework, the four context categories of the PISA 2012 framework have been retained and are used to inform assessment item development. It should be noted that while these contexts are intended to inform item development, there is no expectation that there will be reporting against these contexts.

117. ***Personal*** – Problems classified in the personal context category focus on activities of one's self, one's family or one's peer group. The kinds of contexts that may

be considered personal include (but are not limited to) those involving food preparation, shopping, games, personal health, personal transportation, sports, travel, personal scheduling and personal finance.

118. **Occupational** – Problems classified in the occupational context category are centred on the world of work. Items categorised as occupational may involve (but are not limited to) such things as measuring, costing and ordering materials for building, payroll/accounting, quality control, scheduling/inventory, design/architecture and job-related decision making. Occupational contexts may relate to any level of the workforce, from unskilled work to the highest levels of professional work, although items in the PISA survey must be accessible to 15-year-old students.

119. **Societal** – Problems classified in the societal context category focus on one's community (whether local, national or global). They may involve (but are not limited to) such things as voting systems, public transport, government, public policies, demographics, advertising, national statistics and economics. Although individuals are involved in all of these things in a personal way, in the societal context category, the focus of problems is on the community perspective.

120. **Scientific** – Problems classified in the scientific category relate to the application of mathematics to the natural world and issues and topics related to science and technology. Particular contexts might include (but are not limited to) such areas as weather or climate, ecology, medicine, space science, genetics, measurement and the world of mathematics itself. Items that are intra-mathematical, where all the elements involved belong in the world of mathematics, fall within the scientific context.

121. PISA assessment items are arranged in units that share stimulus material. It is therefore usually the case that all items in the same unit belong to the same context category. Exceptions do arise; for example, stimulus material may be examined from a personal point of view in one item and a societal point of view in another. When an item involves only mathematical constructs without reference to the contextual elements of the unit within which it is located, it is allocated to the context category of the unit. In the unusual case of a unit involving only mathematical constructs and being without reference to any context outside of mathematics, the unit is assigned to the scientific context category.

122. Using these context categories provides the basis for selecting a mix of item contexts and ensures that the assessment reflects a broad range of uses of mathematics, ranging from everyday personal uses to the scientific demands of global problems. Moreover, it is important that each context category be populated with assessment items having a broad range of item difficulties. Given that the major purpose of these context categories is to challenge students in a broad range of problem contexts, each category should contribute substantially to the measurement of mathematical literacy. It should not be the case that the difficulty level of assessment items representing one context category is systematically higher or lower than the difficulty level of assessment items in another category.

123. In identifying contexts that may be relevant, it is critical to keep in mind that a purpose of the assessment is to gauge the use of mathematical content knowledge and skills that students have acquired by age 15. Contexts for assessment items, therefore, are selected in light of relevance to students' interests and lives and the demands that will be placed upon them as they enter society as constructive, engaged and reflective citizens.

National Project Managers from countries participating in the PISA survey are involved in judging the degree of such relevance.

21st Century skills

124. There is increased interest worldwide in what are called 21st century skills and their possible inclusion in educational systems. The OECD has put out a publication focusing on such skills and has sponsored a research project entitled *The Future of Education and Skills: An OECD 2030 Framework* in which some 25 countries are involved in a cross-national study of curriculum including the incorporation of such skills. The project has as its central focus what the curriculum might look like in the future, focusing initially on mathematics.

125. Over the past 15 years or so a number of publications have sought to bring clarity to the discussion and consideration of 21st century skills. A summary of key reports and their conceptualisation of 21st century skills is provided in *PISA 2021 Mathematics: A Broadened Perspective* [EDU/PISA/GB(2017)17]. After careful analysis of these publications the authors recommended that a strong case can be made for the infusion of specific 21st century skills into specific disciplines. For example, it will become increasingly important to teach students at school how to make reasonable arguments and be sure that they are right. The arguments they make should be mathematically rigorous, based on sound theory and strong enough to withstand criticism, and yet, whenever possible, avoid referring to authorities (e.g. 'it says so on the internet'). This is part of the fundamental competence to make independent judgements and take responsibility for them (OECD, 2005[32]). In the social context it is not enough to be right; one must be able and ready to present arguments and to defend them. Learning mathematics, with its clarity of contexts and strong emphasis on logical reasoning and rigour at the appropriate level, is a perfect opportunity to practice and develop the ability for this kind of argumentation.

126. Similarly, in the modern era, it is critical to equip students with tools that they can use to defend themselves from lies. Quite often some fluency in logical reasoning is sufficient; a lie usually hides some hidden contradiction. The alertness of young minds towards possible contradictions can be developed most easily in good classes of mathematics.

127. Using the logic of finding the union between generic 21st century skills and related but subject-matter specific skills that are a natural part of the instruction related to that subject matter results in the following identified eight 21st century skills for inclusion in the PISA 2021 assessment framework. They are:

- Critical thinking
- Creativity
- Research and inquiry
- Self-direction, initiative, and persistence
- Information use
- Systems thinking
- Communication
- Reflection

ASSESSING MATHEMATICAL LITERACY

128. This section outlines the approach taken to implement the elements of the framework described in previous sections into the PISA survey for 2021. This includes the structure of the mathematics component of the PISA survey, the desired distribution of score points for mathematical reasoning and the processes of problem solving; the distribution of score points by content area; a discussion on the range of item difficulties; the structure of the survey instrument; the role of the computer-based assessment of mathematics; the design of the assessment items; and the reporting of levels of mathematical proficiency.

Structure of the PISA 2021 Mathematics Assessment

129. In accordance with the definition of mathematical literacy, assessment items used in any instruments that are developed as part of the PISA survey are set within a context. Items involve the application of important mathematical concepts, knowledge, understandings and skills (mathematical content knowledge) at the appropriate level for 15-year-old students, as described earlier. The framework is used to guide the structure and content of the assessment, and it is important that the survey instrument include an appropriate balance of items reflecting the components of the mathematical literacy framework.

Desired Distribution of Score Points by Mathematical Reasoning and Problem solving process

130. Assessment items in the PISA 2021 mathematics survey can be assigned to either mathematical reasoning or one of three mathematical processes associated with mathematical problem solving. The goal in constructing the assessment is to achieve a balance that provides approximately equal weighting between the two processes that involve making a connection between the real world and the mathematical world (formulating and interpreting/evaluating) and mathematical reasoning and employing which call for students to be able to work on a mathematically formulated problem. While it is true that mathematical reasoning can be observed within the process of formulating, interpreting and employing items will only contribute to one domain.

Table 1. Approximate distribution of score points by domain for PISA 2021

		Percentage of score points in PISA 2021
Mathematical Reasoning		Approximately 25
Mathematical Problem Solving	Formulating Situations Mathematically	Approximately 25
	Employing Mathematical Concepts, Facts, Procedures and Reasoning	Approximately 25
	Interpreting, Applying and Evaluating Mathematical Outcomes	Approximately 25
TOTAL		**100**

131. It is important to note that items in each process category should have a range of difficulty and mathematical demand. This is further addressed in the table of demands for mathematical reasoning and each of the problem solving processes.

Desired Distribution of Score Points by Content Category

132. PISA mathematics items are selected to reflect the mathematical content knowledge described earlier in this framework. The trend items selected for PISA 2021 will be distributed across the four content categories, as shown in Table 2. The goal in constructing the survey is a distribution of items with respect to content category that provides as balanced a distribution of score points as possible, since all of these domains are important for constructive, engaged and reflective citizens.

Table 2. Approximate distribution of score points by content category for PISA 2021

Content category	Percentage of score points in PISA 2021
Change and Relationships	Approximately 25
Space and Shape	Approximately 25
Quantity	Approximately 25
Uncertainty and Data	Approximately 25
TOTAL	**100**

133. It is important to note that items in each content category should have a range of difficulty and mathematical demand.

A Range of Item Difficulties

134. The PISA 2021 mathematical literacy survey includes items with a wide range of difficulties, paralleling the range of abilities of 15-year-old students. It includes items that are challenging for the most able students and items that are suitable for the least able students assessed on mathematical literacy. From a psychometric perspective, a survey that is designed to measure a particular cohort of individuals is most effective and efficient when the difficulty of assessment items matches the ability of the measured subjects. Furthermore, the described proficiency scales that are used as a central part of the reporting of PISA outcomes can only include useful details for all students if the items from which the proficiency descriptions are drawn span the range of abilities described.

135. Table 3 describes the range of actions that are expected of students for mathematical reasoning and each of the problem solving processes. These lists describe

the actions that the items will demand of students. For each category there are a number of items marked with "**" to denote the actions that are expected of the students that will perform at levels 1a, 1b and 1c as well as level 2 of the proficiency scale. Item developers will need to ensure that there are sufficient items at the lower end of the performance scale to allow students at these levels to be able to show what they are capable of.

136. In order to gain useful information for the new lower levels, 1b and 1c, it is vital that context and language do not interfere with the mathematics being assessed. To this end, the context and language must be carefully considered. That said, the items must still be interesting to avoid the possibility that students will simply not attempt the items because it holds no interest.

137. The context for both 1b and 1c level items should be situations that students encounter on a daily basis. Examples of these contexts may include money, temperature, food, time, date, weight, size and distance. All items should be concrete and not abstract. The focus of the item should be mathematical only. The understanding of the context should not interfere with the performance of the item.

138. Equally important, it is to have all items formulated in the simplest possible terms. Sentences should be short and direct. Compound sentences, compound nouns and conditional sentences should be avoided. Vocabulary used in the items must be carefully examined to ensure that students will have a clear understanding of what is being required. In addition, special care will be given to ensure that no extra difficulty is added due to a heavy text load or by a context that is unfamiliar to students based on their cultural background.

139. Items designed for Level 1c should only ask for a single step or operation. However, it is important to note that a single step or operation is not limited to an arithmetical step. This step might be demonstrated by making a selection or identifying some information. Both mathematical reasoning and all of the problem solving processes should be used to measure the mathematical literacy capabilities of students at Levels 1b and 1c.

Table 3. Expected student actions for mathematical reasoning and each of the problem solving processes

Reasoning
** Draw a simple conclusion
** Select an appropriate justification
** Explain why a mathematical result or conclusion does, or does not, make sense given the context of a problem
Represent a problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions
Utilise definitions, rules and formal systems as well as employing algorithms and computational thinking
Explain and defend a justification for the identified or devised representation of a real-world situation
Explain or defend a justification for the processes and procedures or simulations used to determine a mathematical result or solution
Identify the limits of the model used to solve a problem
Understand definitions, rules and formal systems as well as employing algorithms and computational reasoning
Provide a justification for the identified or devised representation of a real-world situation
Provide a justification for the processes and procedures used to determine a mathematical result or solution
Reflect on mathematical arguments, explaining and justifying the mathematical result
Critique the limits of the model used to solve a problem
Interpret a mathematical result back into the real-world context in order to explain the meaning of the results
Explain the relationships between the context-specific language of a problem and the symbolic and formal language needed to represent it mathematically.
Reflect on mathematical arguments, explaining and justifying the mathematical result
Reflect on mathematical solutions and create explanations and arguments that support, refute or qualify a mathematical solution to a contextualised problem
Analyse similarities and differences between a computational model and the mathematical problem that it is modelling
Explain how a simple algorithm works and to detect and correct errors in algorithms and programs

Formulating	Employing	Interpreting
** Select a mathematical description or a representation that describes a problem	** Perform a simple calculation	**Interpret a mathematical result back into the real world context
** Identify the key variables in a model	** Select an appropriate strategy from a list	** Identify whether a mathematical result or conclusion does, or does not, make sense given the context of a problem
** Select a representation appropriate to the problem context	** Implement a given strategy to determine a mathematical solution	** Identify the limits of the model used to solve a problem
Read, decode and make sense of statements, questions, tasks, objects or images to create a model of the situation	** Make mathematical diagrams, graphs, constructions or computing artifacts	Use mathematical tools or computer simulations to ascertain the reasonableness of a mathematical solution and any limits and constraints on that solution, given the context of the problem
Recognise mathematical structure (including regularities, relationships, and patterns) in problems or situations	Understand and utilize constructs based on definitions, rules and formal systems including employing familiar algorithms	Interpret mathematical outcomes in a variety of formats in relation to a situation or use; compare or evaluate two or more representations in relation to a situation
Identify and describe the mathematical aspects of a real-world problem situation including identifying the significant variables	Develop mathematical diagrams, graphs, constructions or computing artifacts and extracting mathematical information from them	Use knowledge of how the real world impacts the outcomes and calculations of a mathematical procedure or model in order to make contextual judgments about how the results should be adjusted or applied
Simplify or decompose a situation or problem in order to make it amenable	Manipulate numbers, graphical and statistical data and information,	Construct and communicate explanations and arguments in the

to mathematical analysis	algebraic expressions and equations, and geometric representations	context of the problem
Recognise aspects of a problem that correspond with known problems or mathematical concepts, facts or procedures	Articulate a solution, showing and/or summarising and presenting intermediate mathematical results	Recognise [demonstrate, interpret, explain] the extent and limits of mathematical concepts and mathematical solutions
Translate a problem into a standard mathematical representation or algorithm	Use mathematical tools, including technology, simulations and computational thinking, to help find exact or approximate solutions	Understand the relationship between the context of the problem and representation of the mathematical solution. Use this understanding to help interpret the solution in context and gauge the feasibility and possible limitations of the solution
Use mathematical tools (using appropriate variables, symbols, diagrams) to describe the mathematical structures and/or relationships in a problem	Make sense of, relate and use a variety of representations when interacting with a problem	
Apply mathematical tools and computing tool to portray mathematical relationships	Switch between different representations in the process of finding solutions	
Identify the constraints, assumptions simplifications in a mathematical model	Use a multi-step procedure leading to a mathematical solution, conclusion or generalization	
	Use an understanding of the context to guide or expedite the mathematical solving process, e.g. working to a context-appropriate level of accuracy	
	Make generalisations based on the results of applying mathematical procedures to find solutions	

Structure of the Survey Instrument

Computer-based Assessment of Mathematics

140. The main mode of delivery for PISA 2021 will be the computer based assessment of mathematics (CBAM). The transition has been anticipated with both the 2015 and 2018 studies moving to computer-based delivery. In order to maintain trends across the studies, both the 2015 and 2018 assessments were computer neutral despite using a computer-based delivery mode. The transition to a full CBAM in 2021 provides a range of opportunities to develop the assessment of mathematical literacy to be better aligned with the evolving nature of mathematics in the modern world, while ensuring backward trends to previous cycles. These opportunities include new item formats (e.g. drag and drop); presenting students with real-world data (such as large, sortable datasets); creating mathematical models or simulations that students can explore by changing the variable values; curve fitting and using the best fit curve to make predictions. In addition to a wider range of question types and mathematical opportunities that the CBAM provides, it also allows for adaptive assessment.

141. The adaptive assessment capability of the CBAM, which was previously implemented in the PISA reading assessment, provides the opportunity of better describing what it is that students at both ends of the performance spectrum are able to do. By providing students with increasing individualised combinations of test units according to their responses and scores to the early units that they respond to,

increasingly detailed information on the performance characteristics of students at both ends of the performance scale is generated.

142. Making use of enhancements offered by computer technology results in assessment items that are more engaging to students, more colourful, and easier to understand. For example, students may be presented with a moving stimulus, representations of three-dimensional objects that can be rotated or more flexible access to relevant information. New item formats, such as those calling for students to ‘drag and drop’ information or use ‘hot spots’ on an image, are designed to engage students, permit a wider range of response types and give a more rounded picture of mathematical literacy. A key challenge is to ensure that these items continue to assess *mathematical literacy* and that interference from domain irrelevant dimensions is kept to a minimum.

143. Investigations show that the mathematical demands of work increasingly occur in the presence of electronic technology so that mathematical literacy and computer use are melded together (Hoyles et al., 2002[33]). For employees at all levels of the workplace, there is now an interdependency between mathematical literacy and the use of computer technology. A key challenge is to distinguish the mathematical demands of a PISA computer-based item from demands unrelated to mathematical competence, such as the information and communications technology (ICT) demands of the item, and the presentation format. Solving PISA items on a computer rather than on paper moves PISA into the reality and the demands of the 21st Century.

144. Questions that seem well suited to the CBAM and the evolving nature of mathematical literacy include:

- Simulation in which a mathematical model has been established and students can change the variable values to explore the impact of the variables to create “an optimal solution”.
- Fitting a curve (by selecting a curve from a limited set of curves provided) to a data set or a geometric image to determine the “best fit” and using the resulting best fit curve to determine the answer to a question about the situation.
- Budgeting situations (e.g. online store) in which the student must select combinations of products to meet achieve a range of objectives within a given budget.
- Purchase simulation in which the student selects from different loan and associates repayment options to purchase an item using a loan and meeting a budget. The challenge in the problem is to understand how the variables interact.
- Problems that include visual coding to achieve a given sequence of actions.

145. Notwithstanding the opportunities that the CBAM presents (described above), it is important that the CBAM remains focussed on assessing mathematical literacy and does not shift to assessing ICT skills. Similarly it is important that the simulations and other questions hinted at above do not become so “noisy” that the mathematical reasoning and problem solving processes are lost.

146. The CBAM must also retain some of the paper version features for example the ability to revisit items already attempted – although in the context of adaptive testing this will of necessity be limited to the unit on which the student is working.

Design of the PISA 2021 Mathematics Items

147. Three item format types are used to assess mathematical literacy in PISA 2021: open constructed-response, closed constructed-response and selected-response (multiple-choice) items.

- Open constructed-response items require a somewhat extended written response from a student. Such items also may ask the student to show the steps taken or to explain how the answer was reached. These items require trained experts to manually code student responses. To facilitate the adaptive assessment feature of the CBAM, it will be necessary to minimize the number of items that rely on trained experts to code the student responses.
- Closed constructed-response items provide a more structured setting for presenting problem solutions, and they produce a student response that can be easily judged to be either correct or incorrect. Often student responses to questions of this type can be coded automatically. The most frequently used closed constructed-responses are single numbers.
- Selected-response items require the choice of one or more responses from a number of response options. Responses to these questions can usually be automatically processed. About equal numbers of each of these item format types are being used to construct the survey instruments.

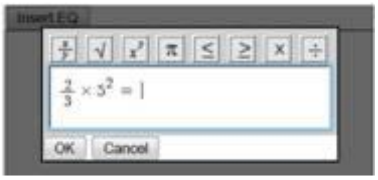
148. The PISA mathematics survey is composed of assessment *units* comprising written stimulus material and other information such as tables, charts, graphs or diagrams, plus one or more items that are linked to this common stimulus material. This format gives students the opportunity to become involved with a context or problem by responding to a series of related items.

149. Items selected for inclusion in the PISA survey represent a broad range of difficulties, to match the wide ability range of students participating in the assessment. In addition, all the major categories of the assessment (the content categories; mathematical reasoning and problem solving process categories and the different context categories and 21st century skills) are represented, to the degree possible, with items of a wide range of difficulties. Item difficulties are established as one of a number of measurement properties in an extensive field trial prior to item selection for the main PISA survey. Items are selected for inclusion in the PISA survey instruments based on their fit with framework categories and their measurement properties.

150. In addition, the level of reading required to successfully engage with an item is considered very carefully in item development and selection. A goal in item development is to make the wording of items as simple and direct as possible. Care is also taken to avoid item contexts that would create a cultural bias, and all choices are checked with national teams. Translation of the items into many languages is conducted very carefully, with extensive back-translation and other protocols.

151. PISA 2021 will include a tool that will allow students to provide typed constructed-response answers and show their work as required for mathematical literacy. The tool allows students to enter both text and numbers. By clicking a button, students can enter a fraction, square root, or exponent. Additional symbols such as pi and greater/less than signs are available, as are operators such as multiplication and division signs. An example is shown in Figure 3 below.

Figure 3. Example of the PISA 2018 editor tool



152. The suite of tools available to students is also expected to include a basic scientific calculator. Operators to be included are addition, subtraction, multiplication and division, as well as square root, pi, parentheses, exponent, square, fraction (y/x), inverse (1/x) and the calculator will be programmed to respect the standard order of operations.

153. Students taking the assessment on paper can have access to a hand-held calculator, as approved for use by 15-year-old students in their respective school systems.

Item Scoring

154. Although the majority of the items are dichotomously scored (that is, responses are awarded either credit or no credit), the open constructed-response items can sometimes involve partial credit scoring, which allows responses to be assigned credit according to differing degrees of "correctness" of responses and or to the extent to which an item has been engaged with or not. It is anticipated that the need for partial credit scoring will be particularly significant for the mathematical reasoning items which will seldom involve the production of single number response but rather responses with one or more elements.

Reporting Proficiency in Mathematics

155. The outcomes of the PISA mathematics survey are reported in a number of ways. Estimates of overall mathematical proficiency are obtained for sampled students in each participating country, and a number of proficiency levels are defined. Descriptions of the degree of mathematical literacy typical of students in each level are also developed. For PISA 2021, the six proficiency levels reported for the overall PISA mathematics in previous cycles will be expanded as follows: Level 1 will be renamed Level 1a, and the table describing the proficiencies will be extended to include Levels 1b and 1c. These additional levels have been added to provide greater granularity of reporting in students performing at the lower end of the proficiency scale.

156. As well as the overall mathematics scale, additional described proficiency scales are developed after the Field Trial and are then reported. These additional scales are for mathematical reasoning and for the three processes of mathematical problem solving: *formulating situations mathematically; employing mathematical concepts, facts, procedures, and reasoning; and interpreting, applying and evaluating mathematical outcomes*.

Mathematical Literacy and the Background Questionnaires

157. Since the first cycle of PISA, student and school context questionnaires have served two interrelated purposes in service of the broader goal of evaluating educational

systems: first, the questionnaires provide a context through which to interpret the PISA results both within and between education systems. Second, the questionnaires aim to provide reliable and valid measurement of additional educational indicators, which can inform policy and research in their own right.

158. Since mathematical literacy is the major domain in the 2021 survey, the background questionnaires are expected to provide not only trend data for the constructs that continue to be assessed, but additionally to provide rich information on the innovations that are evident in the PISA 2021 mathematical literacy framework. In particular it is expected that mathematical literacy will feature prominently in the analysis of the domain-specific contextual constructs as well in a number of the different categories of policy focus that range from individual level variables such a demographics and social and emotional characteristics to school practices, policies and infrastructure (OECD, 2018[34]).

159. Two broad areas of students' attitudes towards mathematics that dispose them to productive engagement in mathematics were identified as being of potential interest as an adjunct to the PISA 2012 mathematics assessment. These are students' interest in mathematics and their willingness to engage in it. It is expected that these will continue to be a focus of the questionnaires in 2021.

160. Interest in mathematics has components related to present and future activity. Relevant questions focus on students' interest in mathematics at school, whether they see it as useful in real life as well as their intentions to undertake further study in mathematics and to participate in mathematics-oriented careers. There is international concern about this area, because in many participating countries there is a decline in the percentage of students who are choosing mathematics related future studies, whereas at the same time there is a growing need for graduates from these areas.

161. Students' willingness to do mathematics is concerned with the attitudes, emotions and self-related beliefs that dispose students to benefit, or prevent them from benefitting, from the mathematical literacy that they have achieved. Students who enjoy mathematical activity and feel confident to undertake it are more likely to use mathematics to think about the situations that they encounter in the various facets of their lives, inside and outside school. The constructs from the PISA survey that are relevant to this area include the emotions of enjoyment, confidence and (lack of) mathematics anxiety, and the self-related beliefs of self-concept and self-efficacy. An analysis of the subsequent progress of young Australians who scored poorly on PISA at age 15 found that those who "recognise the value of mathematics for their future success are more likely to achieve this success, and that includes being happy with many aspects of their personal lives as well as their futures and careers" (Hillman and Thomson, 2010, p. 31[35]). The study recommends that a focus on the practical applications of mathematics in everyday life may help improve the outlook for these low-achieving students.

162. The innovations evident in the PISA 2021 mathematics framework point to at least four areas in which the background questionnaires can provide rich data. These areas are: mathematical reasoning; the role of technology in both doing and teaching mathematics; computational thinking and 21st century skills in the context of mathematics.

163. The foregrounding of mathematical reasoning is the PISA 2021 mathematics framework grounded in the understanding of six "big mathematical ideas": quantity, number systems, abstraction and symbolic representation, the structure of mathematics,

functional relationships between quantities, mathematical modelling, and statistical variance. It is assessed through the focussing of the mathematical contents areas on computer simulation, growth phenomena, conditional decision-making, and geometric approximation and has implications for the background questionnaires which should provide measures to understand students' opportunities to learn these concepts in- and outside of school. In the case of teachers and teaching there is the need to better understand how they see the role of reasoning in mathematics in general and in their teaching and assessment practices in particular. With regard to students the focus is on the extent (or not) of their reasoning and the opportunities that they have to engage in mathematical reasoning both inside the class and outside of it.

164. The PISA 2021 mathematical literacy framework draws attention to the different ways in which technology is both changing the world in which we live (providing greater access to more detailed data in various ways) and changing what it means to do mathematics. Key questions for the background questionnaires include developing a deep understanding of first, how students' experiences of mathematics and doing mathematics are changing (if at all) and second, how classroom pedagogy is evolving on account of the impact that technology is having on our exposure to mathematics and mathematical artefacts and on what it means to do mathematics. In the case of students, it is of interest to better understand how technology is impacting on student performance which could be explored in the *task performance* module of the questionnaire framework. The pedagogical issues could be explored in both the *learning time and curriculum* and *teaching practices* modules.

165. Computational thinking is a rapidly evolving and growing dimension of both mathematics and mathematical literacy. The PISA 2021 mathematical literacy framework illustrates how computational thinking is impacting doing mathematics. The background questionnaires could through the *values and beliefs about learning* and *open-mindedness* modules explore student's experience of the role of computational thinking in doing mathematics.

166. The PISA 2021 mathematical literacy framework introduces a particular set of 21st century skills both as an outcome of and focus for mathematics. The background questionnaires could productively examine both whether or not mathematics is contributing to the development of these skills and if teaching practices are focussing on them. In particular, the *learning time and curriculum* module could explore whether or not these skills appear in the enacted curriculum.

167. Previously the student questionnaire has included items related to the *opportunity to learn*. There have been items that deal with student experience with applied mathematics problems of various types, student familiarity with mathematical concepts by name and student experience in class or tests with PISA style items. These measures allow deeper analysis of the PISA results and the topic could be covered in 2021 as well.

168. The results of the PISA 2021 survey will provide important information for educational policy makers in the participating countries about both the achievement-related and attitude-related outcomes of schooling. By combining information from the PISA assessment of mathematical literacy and the survey information on attitudes, emotions and beliefs that predispose students to use their mathematical literacy as well as the impact of the four developments described above, a more complete picture will emerge.

SUMMARY

169. The PISA 2021 mathematical literacy framework while maintaining coherence with the previous mathematical literacy frameworks acknowledges that the world is ever changing and with it the demand for mathematically literate citizens to reason mathematically rather than reproducing mathematical techniques as routines,

170. The aim of PISA with regard to mathematical literacy is to develop indicators that show how effectively countries are preparing students to use mathematics in the everyday aspect of their personal, civic and professional lives, as constructive, engaged and reflective 21st century citizens. To achieve this, PISA has developed a definition of mathematical literacy and an assessment framework that reflects the important components of this definition.

171. The mathematics assessment items selected for inclusion in PISA 2021, based on this definition and framework, are intended to reflect a balance between mathematical reasoning, problem solving processes, mathematical content and contexts.

172. The CBAM to be used from 2021 provides problems in a variety of item formats with varying degrees of built-in guidance and structure and a range of formats retaining throughout an emphasis on authentic problems that require students to reason and demonstrate their thinking.

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